

ON THE EXTENSIONS OF BARLOW-PROSCHAN IMPORTANCE INDEX AND SYSTEM SIGNATURE TO DEPENDENT LIFETIMES

JEAN-LUC MARICHAL AND PIERRE MATHONET

ABSTRACT. For a coherent system the Barlow-Proschan importance index, defined when the component lifetimes are independent, measures the probability that the failure of a given component causes the system to fail. Iyer (1992) extended this concept to the more general case when the component lifetimes are jointly absolutely continuous but not necessarily independent. Assuming only that the joint distribution of component lifetimes has no ties, we give an explicit expression for this extended index in terms of the discrete derivatives of the structure function and provide an interpretation of it as a probabilistic value, a concept introduced in game theory. This enables us to interpret Iyer's formula in this more general setting. We also discuss the analogy between this concept and that of system signature and show how it can be used to define a symmetry index for systems.

1. INTRODUCTION

Consider an n -component system $S = (n, \phi, F)$, where ϕ denotes the associated structure function $\phi: \{0, 1\}^n \rightarrow \{0, 1\}$ (which expresses the state of the system in terms of the states of its components) and F denotes the joint c.d.f. of the component lifetimes X_1, \dots, X_n , that is,

$$F(t_1, \dots, t_n) = \Pr(X_1 \leq t_1, \dots, X_n \leq t_n), \quad t_1, \dots, t_n \geq 0.$$

We assume that the system is *semicoherent*, i.e., the structure function ϕ is nondecreasing in each variable and satisfies the conditions $\phi(\mathbf{0}) = 0$ and $\phi(\mathbf{1}) = 1$.

An *importance index* was introduced in 1975 by Barlow and Proschan [1] for systems whose components have continuous and independent lifetimes as the n -tuple \mathbf{I}_{BP} whose k th coordinate $I_{\text{BP}}^{(k)}$ is the probability that the failure of component k causes the system to fail. In mathematical terms,

$$(1) \quad I_{\text{BP}}^{(k)} = \Pr(T = X_k), \quad k \in [n],$$

where T denotes the system lifetime and $[n] = \{1, \dots, n\}$.

When the components have i.i.d. lifetimes, this index reduces to the n -tuple $\mathbf{b} = (b_1, \dots, b_n)$, where

$$(2) \quad b_k = \sum_{A \subseteq [n] \setminus \{k\}} \frac{1}{n \binom{n-1}{|A|}} \Delta_k \phi(A) = \frac{1}{n} \sum_{j=0}^{n-1} \frac{1}{\binom{n-1}{j}} \sum_{\substack{A \subseteq [n] \setminus \{k\} \\ |A|=j}} \Delta_k \phi(A)$$

Date: September 22, 2011.

2010 Mathematics Subject Classification. 62N05, 90B25, 94C10.

Key words and phrases. Coherent system, component importance, Barlow-Proschan index, dependent lifetimes, system signature.

and $\Delta_k \phi(A) = \phi(A \cup \{k\}) - \phi(A \setminus \{k\})$.¹ Thus, in the i.i.d. case the probability (1) does not depend of the c.d.f. F . Due to this feature it is sometimes referred to as a *structural importance*. Note that the expression in the right-hand side of (2) was actually defined in 1953 in cooperative game theory where it is known as the Shapley-Shubik value [9, 10].

The concept of *signature*, which reveals a strong analogy with that of Barlow-Proschan importance index (1), was introduced in 1985 by Samaniego [7] (see also [8]) for systems whose components have continuous and i.i.d. lifetimes as the n -tuple $\mathbf{s} = (s_1, \dots, s_n)$, where s_k is the probability that the k th component failure causes the system to fail. That is,

$$s_k = \Pr(T = X_{k:n}),$$

where $X_{k:n}$ denotes the k th smallest lifetime, i.e., the k th order statistic obtained by rearranging the variables X_1, \dots, X_n in ascending order of magnitude.

Boland [2] showed that s_k can be explicitly written in the form

$$(3) \quad s_k = \sum_{\substack{A \subseteq [n] \\ |A|=n-k+1}} \frac{1}{\binom{n}{|A|}} \phi(A) - \sum_{\substack{A \subseteq [n] \\ |A|=n-k}} \frac{1}{\binom{n}{|A|}} \phi(A).$$

Just as for the probability $\Pr(T = X_k)$, in the i.i.d. case the probability $\Pr(T = X_{k:n})$ does not depend on the c.d.f. F . Thus \mathbf{s} can be regarded as the *structural signature*.

Example 1. For a system made up of three serially connected components with i.i.d. lifetimes, we have $\mathbf{s} = (1, 0, 0)$ and $\mathbf{b} = (1/3, 1/3, 1/3)$.

Iyer [3] extended the Barlow-Proschan index to the general dependent case where the c.d.f. F is absolutely continuous. In this setting the index \mathbf{I}_{BP} may depend not only on the structure function ϕ but also on the c.d.f. F . Specifically, starting from the multilinear form of ϕ ,

$$\phi(\mathbf{x}) = \sum_{A \subseteq [n]} m_\phi(A) \prod_{i \in A} x_i,$$

where $m_\phi: 2^{[n]} \rightarrow \mathbb{R}$ is the Möbius transform of ϕ , defined by

$$m_\phi(A) = \sum_{B \subseteq A} (-1)^{|A|-|B|} \phi(B),$$

Iyer obtained the integral formula

$$(4) \quad I_{\text{BP}}^{(k)} = \sum_{A \subseteq [n] \setminus \{k\}} m_\phi(A \cup \{k\}) \int_0^\infty \frac{d}{dt_k} \Pr\left(X_k \leq t_k \text{ and } t < \min_{j \in A} X_j\right)_{t_k=t} dt.$$

The concept of signature was also extended to the general case of dependent lifetimes; see [4] (see also [6] for an earlier work). Denoting this “extended” signature by the n -tuple $\mathbf{p} = (p_1, \dots, p_n)$, where $p_k = \Pr(T = X_{k:n})$, the authors [4] proved that, if F is absolutely continuous (actually the assumption that there are no ties among the component lifetimes is sufficient), then

$$(5) \quad p_k = \sum_{|A|=n-k+1} q(A) \phi(A) - \sum_{|A|=n-k} q(A) \phi(A),$$

¹As usual, we identify Boolean vectors $\mathbf{x} \in \{0, 1\}^n$ and subsets $A \subseteq [n]$ by setting $x_i = 1$ if and only if $i \in A$. We thus use the same symbol to denote both a function $f: \{0, 1\}^n \rightarrow \mathbb{R}$ and its corresponding set function $f: 2^{[n]} \rightarrow \mathbb{R}$, interchangeably.

where the function $q: 2^{[n]} \rightarrow [0, 1]$, called the *relative quality function* associated with F , is defined by

$$q(A) = \Pr \left(\max_{i \notin A} X_i < \min_{j \in A} X_j \right).^2$$

Thus (5) is the non-i.i.d. extension of (3). Note also that the function q has the immediate property

$$(6) \quad \sum_{|A|=k} q(A) = 1, \quad k \in [n].$$

In this paper, assuming only that F has no ties, we give an alternative expression for $I_{\text{BP}}^{(k)}$ as a weighted arithmetic mean over $A \subseteq [n] \setminus \{k\}$ of $\Delta_k \phi(A)$ and whose coefficients depend only on F (Theorem 3), thus providing the analog of (5) for the Barlow-Proschan index. This enables us to retrieve and interpret Iyer's formula (4) in this more general setting (Corollary 5). We give necessary and sufficient conditions on F for $I_{\text{BP}}^{(k)}$ to always reduce to (2) regardless of the structure function considered (Proposition 8). We also provide explicit expressions for the coefficient of $\Delta_k \phi(A)$ in the general continuous and independent continuous cases (Propositions 9 and 10) and examine the special case of independent Weibull lifetimes, which includes the exponential model (Corollary 11). Finally, we show how the Barlow-Proschan index can be used to measure a symmetry degree of any system (Section 4).

2. EXPLICIT EXPRESSIONS

Throughout we assume that the joint c.d.f. F of the lifetimes has *no ties*, i.e., we have $\Pr(X_i = X_j) = 0$ for every $i \neq j$. For every $k \in [n]$, we define the function $q_k: 2^{[n] \setminus \{k\}} \rightarrow [0, 1]$ as

$$q_k(A) = \Pr \left(\max_{i \notin A \cup \{k\}} X_i < X_k < \min_{j \in A} X_j \right).$$

Thus defined, $q_k(A)$ is the probability that the components that are better than component k are precisely those in A . It then follows immediately that

$$(7) \quad \sum_{A \subseteq [n] \setminus \{k\}} q_k(A) = 1, \quad k \in [n].$$

We also observe that

$$(8) \quad q(A) = \sum_{k \notin A} q_k(A), \quad A \neq [n],$$

$$(9) \quad q(A) = \sum_{k \in A} q_k(A \setminus \{k\}), \quad A \neq \emptyset.$$

Moreover, $q_k(\emptyset) = q(\{k\})$ is the probability that component k is the best component, while $q_k([n] \setminus \{k\}) = q([n] \setminus \{k\})$ is the probability that component k is the worst component.

Proposition 2. *If the variables X_1, \dots, X_n are exchangeable, then*

$$(10) \quad q_k(A) = \frac{1}{(n-|A|)\binom{n}{|A|}} = \frac{1}{n\binom{n-1}{|A|}}$$

for every $k \in [n]$ and every $A \subseteq [n] \setminus \{k\}$.

²Thus $q(A)$ is the probability that the best $|A|$ components are precisely those in A .

Proof. Since X_1, \dots, X_n are exchangeable, by (6) we have $q(A) = 1/\binom{n}{|A|}$ (see [4]). Then, by definition of $q_k(A)$, we have

$$q_k(A) = \Pr\left(\max_{i \notin A} X_i = X_k \mid \max_{i \notin A} X_i < \min_{j \in A} X_j\right) q(A) = \frac{1}{(n - |A|)\binom{n}{|A|}},$$

which completes the proof. \square

We now give an expression for $I_{\text{BP}}^{(k)}$ in terms of the functions q_k and $\Delta_k \phi$. This expression, given in (11) below, clearly extends (2) just as formula (5) extends (3).

Theorem 3. *For every $k \in [n]$, we have*

$$(11) \quad I_{\text{BP}}^{(k)} = \sum_{A \subseteq [n] \setminus \{k\}} q_k(A) \Delta_k \phi(A) = \sum_{A \subseteq [n]} (-1)^{|\{k\} \setminus A|} q_k(A \setminus \{k\}) \phi(A).$$

Proof. Denote the set of permutations on $[n]$ by \mathfrak{S}_n . For every fixed $\sigma \in \mathfrak{S}_n$ we must have

$$(12) \quad \Pr(T = X_k \mid X_{\sigma(1)} < \dots < X_{\sigma(n)}) = \phi(\{\sigma(j), \dots, \sigma(n)\}) - \phi(\{\sigma(j+1), \dots, \sigma(n)\}),$$

where $j = \sigma^{-1}(k)$. Indeed, the left-hand expression of (12) takes its values in $\{0, 1\}$ and is exactly 1 if and only if $\{\sigma(1), \dots, \sigma(j-1)\}$ is not a cut set and $\{\sigma(1), \dots, \sigma(j)\}$ is a cut set.³ Equivalently,

$$\phi(\{\sigma(j), \dots, \sigma(n)\}) = 1 \quad \text{and} \quad \phi(\{\sigma(j+1), \dots, \sigma(n)\}) = 0.$$

Now, by combining (12) with the law of total probability, we get

$$I_{\text{BP}}^{(k)} = \sum_{\sigma \in \mathfrak{S}_n} \left(\phi(\{\sigma(\sigma^{-1}(k)), \dots, \sigma(n)\}) - \phi(\{\sigma(\sigma^{-1}(k)+1), \dots, \sigma(n)\}) \right) \Pr(X_{\sigma(1)} < \dots < X_{\sigma(n)}).$$

Grouping the terms for which $\{\sigma(\sigma^{-1}(k)+1), \dots, \sigma(n)\}$ is a fixed set A and then summing over A , we obtain

$$I_{\text{BP}}^{(k)} = \sum_{A \subseteq [n] \setminus \{k\}} (\phi(A \cup \{k\}) - \phi(A)) \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \{\sigma(n-|A|+1), \dots, \sigma(n)\} = A \\ \sigma(n-|A|) = k}} \Pr(X_{\sigma(1)} < \dots < X_{\sigma(n)}).$$

The result then follows since the inner sum is exactly $q_k(A)$. The second expression in (11) follows immediately from the first one. \square

Example 4. Assume that ϕ defines a k -out-of- n structure, that is, $\phi(\mathbf{x}) = x_{k:n}$, where $x_{k:n}$ is the k th order statistic of the variables x_1, \dots, x_n . In this case we have $\phi(A) = 1$ if and only if $|A| \geq n - k + 1$ and hence, for every $\ell \in [n]$, we have $\Delta_\ell \phi(A) = 1$ if and only if $|A| = n - k$. By Theorem 3,

$$I_{\text{BP}}^{(\ell)} = \sum_{\substack{A \subseteq [n] \setminus \{\ell\} \\ |A| = n-k}} q_\ell(A)$$

is the probability that component ℓ has the k th smallest lifetime. This result was expected since $I_{\text{BP}}^{(\ell)} = \Pr(X_\ell = T) = \Pr(X_\ell = X_{k:n})$ by definition.

³Recall that a subset $K \subseteq [n]$ of components is a *cut set* for the function ϕ if $\phi([n] \setminus K) = 0$.

Formula (11) provides an explicit expression for the Barlow-Proschan index under the sole assumption that F has no ties, which leads to easy interpretations and computations and reveals an interesting analogy with the concept of signature. Combining this formula with (7) also shows that \mathbf{I}_{BP} is a *probabilistic value*, as defined in game theory by Weber [11]. Moreover, \mathbf{I}_{BP} is *efficient* in the sense that $\sum_{k=1}^n I_{\text{BP}}^{(k)} = 1$.

We now use (11) to derive Iyer's formula (4) in an interpretable form and without the absolute continuity assumption. For every $k \in [n]$, we define the function $r_k: 2^{[n] \setminus \{k\}} \rightarrow [0, 1]$ as

$$r_k(A) = \Pr\left(X_k < \min_{j \in A} X_j\right).$$

That is, $r_k(A)$ is the probability that the components in A are better than component k . We then have

$$(13) \quad r_k(A) = \sum_{\substack{B \subseteq [n] \setminus \{k\} \\ B \supseteq A}} q_k(B),$$

which can be inverted into

$$(14) \quad q_k(A) = \sum_{\substack{B \subseteq [n] \setminus \{k\} \\ B \supseteq A}} (-1)^{|B| - |A|} r_k(B).$$

Corollary 5. *For every $k \in [n]$, we have*

$$I_{\text{BP}}^{(k)} = \sum_{A \subseteq [n] \setminus \{k\}} r_k(A) m_\phi(A \cup \{k\}).$$

Proof. Using the inverse Möbius transform $\phi(A) = \sum_{B \subseteq A} m_\phi(B)$ in the right-hand side of (11) and then permuting the resulting sums, we obtain

$$I_{\text{BP}}^{(k)} = \sum_{B \subseteq [n]} m_\phi(B) \sum_{A \supseteq B} (-1)^{|\{k\} \setminus A|} q_k(A \setminus \{k\}).$$

Noticing that the inner sum vanishes whenever $B \not\supseteq k$ and setting $A' = A \setminus \{k\}$ and $B' = B \setminus \{k\}$, we obtain

$$I_{\text{BP}}^{(k)} = \sum_{B' \subseteq [n] \setminus \{k\}} m_\phi(B' \cup \{k\}) \sum_{\substack{A' \subseteq [n] \setminus \{k\} \\ A' \supseteq B'}} q_k(A').$$

We then conclude by (13). \square

Just as for the Barlow-Proschan index, the signature \mathbf{p} also has an interesting expression in terms of the Möbius transform of ϕ .

Proposition 6. *For every $k \in [n]$, we have*

$$p_k = \sum_{A \subseteq [n]} m_\phi(A) \Pr\left(X_{k:n} = \min_{j \in A} X_j\right),$$

where $\Pr(X_{k:n} = \min_{j \in A} X_j)$ is the probability that the k th failure is that of the worst component in A .

Proof. Using the inverse Möbius transform, for every $k \in [n]$ we have

$$\sum_{|B|=k} q(B) \phi(B) = \sum_{|B|=k} q(B) \sum_{A \subseteq B} m_\phi(A) = \sum_{A \subseteq [n]} m_\phi(A) \sum_{\substack{B \supseteq A \\ |B|=k}} q(B),$$

where the inner sum is the probability $\Pr(X_{n-k:n} < \min_{j \in A} X_j)$ that the components in A be among the best k components. We then conclude by (5). \square

Remark 1. The proof of Proposition 6 shows that

$$\Pr(X_{k:n} = \min_{j \in A} X_j) = \sum_{\substack{B \supseteq A \\ |B|=n-k+1}} q(B) - \sum_{\substack{B \supseteq A \\ |B|=n-k}} q(B)$$

is exactly the signature p_k of the system obtained by transforming the current structure function into $\phi(\mathbf{x}) = \prod_{i \in A} x_i$.⁴ It also shows that the *tail signature* [4], defined by $\Pr(T > X_{k:n}) = \sum_{j=k+1}^n p_j$, has the Möbius representation

$$\Pr(T > X_{k:n}) = \sum_{|A|=n-k} q(A) \phi(A) = \sum_{A \subseteq [n]} m_\phi(A) \Pr(X_{k:n} < \min_{j \in A} X_j).$$

The next corollary, which follows immediately from Proposition 2 and Theorem 3, gives a sufficient condition on F for the equality $\mathbf{I}_{\text{BP}} = \mathbf{b}$ to hold regardless of the structure function considered.

Corollary 7. *If the variables X_1, \dots, X_n are exchangeable, then $\mathbf{I}_{\text{BP}} = \mathbf{b}$.*

We now give necessary and sufficient conditions on F for the equality $\mathbf{I}_{\text{BP}} = \mathbf{b}$ to hold for every structure function.

A system is said to be *coherent* if it is semicoherent and its structure function ϕ has only *essential* variables, i.e., for every $k \in [n]$, there exists $\mathbf{x} \in \{0, 1\}^n$ such that $\phi(\mathbf{x})|_{x_k=0} \neq \phi(\mathbf{x})|_{x_k=1}$.⁵ Let Φ_n (resp. Φ'_n) denote the family of n -variable structure functions corresponding to coherent (resp. semicoherent) systems.

Proposition 8. *The equality $\mathbf{I}_{\text{BP}} = \mathbf{b}$ holds for every $\phi \in \Phi_n$ (or equivalently, for every $\phi \in \Phi'_n$) if and only if (10) holds for every $k \in [n]$ and every $A \subseteq [n] \setminus \{k\}$.*

Proof. We can assume that $n \geq 3$ (the cases $n = 1$ and $n = 2$ can be checked easily). Using (11) and (2), we see that the identity $\mathbf{I}_{\text{BP}} = \mathbf{b}$ can be written as

$$\sum_{A \subseteq [n]} (-1)^{|\{k\} \setminus A|} q_k(A \setminus \{k\}) \phi(A) = \sum_{A \subseteq [n]} (-1)^{|\{k\} \setminus A|} \frac{1}{n \binom{n-1}{|A \setminus \{k\}|}} \phi(A).$$

It was shown in [5] that for any function $\lambda: 2^{[n]} \rightarrow \mathbb{R}$ we have

$$\sum_{A \subseteq [n]} \lambda(A) \phi(A) = 0, \quad \text{for every } \phi \in \Phi_n \text{ (or every } \phi \in \Phi'_n)$$

if and only if $\lambda(A) = 0$ for all $A \neq \emptyset$. Therefore we have $\mathbf{I}_{\text{BP}} = \mathbf{b}$ for every $\phi \in \Phi_n$ (or every $\phi \in \Phi'_n$) if and only if $q_k(A \setminus \{k\}) = 1/(n \binom{n-1}{|A \setminus \{k\}|})$ for every $k \in [n]$ and every $A \neq \emptyset$. This completes the proof. \square

We observe that $q_k(A)$ has the form (10), as given in Propositions 2 and 8, if and only if the map $(k, A) \mapsto q_k(A)$ is symmetric in the sense that

$$q_{\sigma(k)}(\sigma(A)) = q_k(A)$$

⁴This fact also follows immediately from the identity $p_k = \Pr(T = X_{k:n})$ since this modified system has lifetime $T = \min_{j \in A} X_j$.

⁵In other words, every component of the system is relevant.

for every $k \in [n]$, every $A \subseteq [n] \setminus \{k\}$, and every permutation σ on $[n]$. Indeed, by (6) and (8), for any $j \in [n-1]$ we have

$$\sum_{|A|=j} \sum_{k \notin A} q_k(A) = \sum_{|A|=j} q(A) = 1.$$

The identity (10) then follows from the symmetry of the map $(k, A) \mapsto q_k(A)$.⁶

The “signature” version of Proposition 8 can be stated as follows (see [5]). For $n \geq 3$ (resp. $n \geq 2$), the equality $\mathbf{p} = \mathbf{s}$ holds for every $\phi \in \Phi_n$ (resp. every $\phi \in \Phi'_n$) if and only if $q(A) = 1/\binom{n}{|A|}$ for every $A \subseteq [n]$ (i.e., q is a symmetric function).

3. CONTINUOUS AND INDEPENDENT CONTINUOUS LIFETIMES

We now assume that the component lifetimes are absolutely continuous with p.d.f. f . This assumption enables us to derive explicit integral formulas for $q_k(A)$ and $r_k(A)$.

Proposition 9. *For absolutely continuous lifetimes, we have*

$$(15) \quad q_k(A) = \int_0^\infty \int_{]0, t_k[\setminus ([n] \setminus (A \cup \{k\}))} \int_{]t_k, \infty[^A} f(\mathbf{t}) d\mathbf{t}_A d\mathbf{t}_{[n] \setminus (A \cup \{k\})} dt_k$$

and

$$(16) \quad r_k(A) = \int_0^\infty \int_{]0, \infty[\setminus ([n] \setminus (A \cup \{k\}))} \int_{]t_k, \infty[^A} f(\mathbf{t}) d\mathbf{t}_A d\mathbf{t}_{[n] \setminus (A \cup \{k\})} dt_k$$

for every $k \in [n]$ and every $A \subseteq [n] \setminus \{k\}$.

Proof. By definition we have $q_k(A) = \Pr(E)$, where E is the event

$$\max_{i \notin A \cup \{k\}} X_i < X_k < \min_{j \in A} X_j,$$

which can be described by the set

$$\{(t_1, \dots, t_n) \in]0, \infty[^n : t_i < t_k \ \forall i \notin A \cup \{k\}, t_k < t_j \ \forall j \in A\}.$$

Formula (15) then follows by integrating the p.d.f. of this event. Formula (16) can be established similarly. \square

We now consider the special case of independent and absolutely continuous lifetimes X_1, \dots, X_n , each X_i having p.d.f. f_i and c.d.f. F_i , with $F_i(0) = 0$. The following immediate result shows how formulas (15) and (16) can be simplified. Similar formulas for $q(A)$ can be found in [4].

Proposition 10. *For independent and absolutely continuous lifetimes, we have*

$$(17) \quad q_k(A) = \int_0^\infty f_k(t) \prod_{i \in A} \overline{F}_i(t) \prod_{i \notin A \cup \{k\}} F_i(t) dt$$

and

$$(18) \quad r_k(A) = \int_0^\infty f_k(t) \prod_{i \in A} \overline{F}_i(t) dt$$

for every $k \in [n]$ and every $A \subseteq [n] \setminus \{k\}$, where $\overline{F}_i(t) = 1 - F_i(t)$.

Using (18) and then (14) leads immediately to the following corollary.

⁶We note that this result provides an alternative proof of Proposition 2.

Corollary 11. *For independent Weibull lifetimes, with $F_i(t) = 1 - e^{-(\lambda_i t)^\alpha}$, we have*

$$(19) \quad r_k(A) = \frac{\lambda_\alpha(\{k\})}{\lambda_\alpha(A \cup \{k\})}$$

and

$$(20) \quad q_k(A) = \sum_{\substack{B \subseteq [n] \setminus \{k\} \\ B \supseteq A}} (-1)^{|B| - |A|} \frac{\lambda_\alpha(\{k\})}{\lambda_\alpha(B \cup \{k\})}$$

for every $k \in [n]$ and every $A \subseteq [n] \setminus \{k\}$, where $\lambda_\alpha(A) = \sum_{i \in A} \lambda_i^\alpha$.

We observe that, under the assumptions of Corollary 11, by (19) the ratio

$$\frac{\lambda_\alpha(\{k\})}{\lambda_\alpha([n])} = r_k([n] \setminus \{k\}) = q_k([n] \setminus \{k\})$$

is exactly the probability that X_k is the shortest lifetime.

4. A SYMMETRY INDEX FOR SYSTEMS

A natural concept of symmetry for systems can be defined as follows. We say that a semicoherent system $S = (n, \phi, F)$ is *symmetric* if it has a uniform Barlow-Proschan index, i.e., $\mathbf{I}_{\text{BP}} = (1/n, \dots, 1/n)$.

Example 12. For a system made up of n serially connected components, we have $I_{\text{BP}}^{(k)} = q_k([n] \setminus \{k\}) = q([n] \setminus \{k\})$ for every $k \in [n]$. The system is then symmetric if and only if $q([n] \setminus \{k\})$ is independent of k .

Since a system is rarely symmetric in the non-i.i.d. case, it is natural to define an index measuring the “symmetry degree” of the system.

Recall from probability theory that the uniformity of a probability distribution $\mathbf{w} = (w_1, \dots, w_n)$ over $[n]$ can be measured through the concept of *normalized Shannon entropy*

$$H(\mathbf{w}) = -\frac{1}{\ln n} \sum_{k=1}^n w_k \ln(w_k),$$

with the convention that $0 \ln 0 = 0$. It is well known that $H(\mathbf{w})$ is maximum ($H(\mathbf{w}) = 1$) if and only if \mathbf{w} is the uniform distribution $\mathbf{w}^* = (1/n, \dots, 1/n)$ and minimum ($H(\mathbf{w}) = 0$) if and only if $w_k = 1$ for some $k \in [n]$ and $w_i = 0$ for all $i \neq k$ (Dirac measure). Moreover, for any probability distribution $\mathbf{w} \neq \mathbf{w}^*$, the expression $H(\mathbf{w}_\lambda)$, where $\mathbf{w}_\lambda = \mathbf{w} + \lambda(\mathbf{w}^* - \mathbf{w})$, strictly increases as the parameter λ increases from 0 to 1. Thus, the number $H(\mathbf{w})$, which lies in the interval $[0, 1]$, measures an uniformity (evenness) degree of the probability distribution \mathbf{w} .

On the basis of these observations, we define the *symmetry index* of S as the number $H(\mathbf{I}_{\text{BP}})$, that is, the normalized Shannon entropy of \mathbf{I}_{BP} .

In the next two propositions we provide conditions under which the entropies $H(\mathbf{I}_{\text{BP}})$ and $H(\mathbf{p})$ reach their extreme values. These conditions reveal that the tuples \mathbf{I}_{BP} and \mathbf{p} in a sense may have opposite behaviors.

Proposition 13. *Let $S = (n, \phi, F)$ be a semi-coherent system.*

- (i) *If the functions q_j ($j = 1, \dots, n$) are strictly positive, then $H(\mathbf{I}_{\text{BP}}) = 0$ if and only if $\phi(\mathbf{x}) = x_k$ for some $k \in [n]$ (i.e., exactly one component of S is relevant).*

- (ii) If the function q is strictly positive, then $H(\mathbf{p}) = 0$ if and only if $\phi(\mathbf{x}) = x_{k:n}$ for some $k \in [n]$ (i.e., S is a k -out-of- n system).

Proof. Let us prove (i). Suppose first that $H(\mathbf{I}_{\text{BP}}) = 0$. This means that $I_{\text{BP}}^{(k)} = 1$ for some $k \in [n]$. Equivalently, by (11) we have

$$\sum_{A \subseteq [n] \setminus \{j\}} q_j(A) \Delta_j \phi(A) = \begin{cases} 1, & \text{if } j = k \\ 0, & \text{otherwise.} \end{cases}$$

By (7) and due to the positivity of q_j , this means that $\Delta_j \phi(A) = 1$, if $j = k$, and 0, otherwise. Equivalently, $\phi(A) = 1$ if and only if $k \in A$. Thus, $\phi(\mathbf{x}) = x_k$. The converse implication immediately follows from (7) and (11).

Let us prove (ii). Suppose first that $H(\mathbf{p}) = 0$. This means that $p_k = 1$ for some $k \in [n]$. Equivalently, by (5) we have,

$$\sum_{|A|=j} q(A) \phi(A) = \begin{cases} 1, & \text{if } j \geq n - k + 1 \\ 0, & \text{otherwise.} \end{cases}$$

By (6) and due to the positivity of q , this condition means that $\phi(A) = 1$, if $|A| \geq n - k + 1$, and 0, otherwise. Thus, $\phi(\mathbf{x}) = x_{k:n}$. The converse implication immediately follows from (5) and (6). \square

Proposition 14. Let $S = (n, \phi, F)$ be a semi-coherent system.

- (i) If the functions q_j ($j = 1, \dots, n$) have the form (10) and $\phi(\mathbf{x}) = x_{k:n}$ for some $k \in [n]$, then $H(\mathbf{I}_{\text{BP}}) = H(\mathbf{b}) = 1$.
(ii) If the function q is symmetric and $\phi(\mathbf{x}) = x_k$ for some $k \in [n]$, then $H(\mathbf{p}) = H(\mathbf{s}) = 1$.

Proof. Let us prove (i). Since $\phi(\mathbf{x}) = x_{k:n}$ for some k , we have $\Delta_j \phi(A) = 1$ if and only if $|A| \geq n - k$. It follows that

$$I_{\text{BP}}^{(j)} = \binom{n-1}{n-k} \frac{1}{n \binom{n-1}{n-k}} = \frac{1}{n}$$

and hence $H(\mathbf{I}_{\text{BP}}) = 1$. We also have $\mathbf{I}_{\text{BP}} = \mathbf{b}$ by Proposition 8.

Let us prove (ii). Since $\phi(\mathbf{x}) = x_k$ for some k , we have $\phi(A) = 1$ if and only if $k \in A$. It follows that

$$p_j = \sum_{|A|=n-j+1} \frac{1}{\binom{n}{|A|}} \phi(A) - \sum_{|A|=n-j} \frac{1}{\binom{n}{|A|}} \phi(A) = \frac{\binom{n-1}{n-j}}{\binom{n}{n-j+1}} - \frac{\binom{n-1}{n-j-1}}{\binom{n}{n-j}} = \frac{1}{n}$$

for $j < n$ (and also for $j = n$) and hence $H(\mathbf{p}) = 1$. We also have $\mathbf{p} = \mathbf{s}$ (see the last paragraph in Section 2). \square

Remark 2. Put in other words, Proposition 13 states that

- (i) Under positiveness of the functions q_k , the mass of \mathbf{I}_{BP} is concentrated on exactly one coordinate (\mathbf{I}_{BP} is a Dirac measure) if and only if exactly one component of the system is relevant.
(ii) Under positiveness of the function q , the mass of \mathbf{p} is concentrated on exactly one coordinate (\mathbf{p} is a Dirac measure) if and only if the system is of k -out-of- n type for some k .

Similarly, Proposition 14 states that

- (i) For a k -out-of- n system, if the functions q_j ($j = 1, \dots, n$) have the form (10), then \mathbf{I}_{BP} is uniform.
- (ii) For a system with only one relevant component, if the function q is symmetric, then \mathbf{p} is uniform.

ACKNOWLEDGMENTS

This research is supported by the internal research project F1R-MTH-PUL-09MRDO of the University of Luxembourg.

REFERENCES

- [1] R.E. Barlow and F. Proschan. Importance of system components and fault tree events. *Stochastic Processes and Their Applications*, 3:153–172, 1975.
- [2] P. J. Boland. Signatures of indirect majority systems. *J. Appl. Prob.*, 38:597–603, 2001.
- [3] S. Iyer. The Barlow-Proschan importance and its generalizations with dependent components. *Stochastic Process. Appl.*, 42:353–359, 1992.
- [4] J.-L. Marichal and P. Mathonet. Extensions of system signatures to dependent lifetimes: Explicit expressions and interpretations. *J. Multivariate Analysis*, 102(5):931–936, 2011.
- [5] J.-L. Marichal, P. Mathonet, and T. Waldhauser. On signature-based expressions of system reliability. *J. Multivariate Analysis*, 102(10): 1410–1416, 2011.
- [6] J. Navarro, F.J. Samaniego, N. Balakrishnan, and D. Bhattacharya. On the application and extension of system signatures in engineering reliability. *Naval Research Logistics*, 55:313–327, 2008.
- [7] F.J. Samaniego. On closure of the IFR class under formation of coherent systems. *IEEE Trans. Reliability Theory*, 34:69–72, 1985.
- [8] F.J. Samaniego. *System signatures and their applications in engineering reliability*. Int. Series in Operations Research & Management Science, 110. New York: Springer, 2007.
- [9] L. Shapley. A value for n -person games. In *Contributions to the Theory of Games II (Annals of Mathematics Studies 28)*, pages 307–317. Princeton University Press, 1953.
- [10] L. Shapley and M. Shubik. A method for evaluating the distribution of power in a committee system. *American Political Science Review*, 48:787–792, 1954.
- [11] R.J. Weber. *Probabilistic values for games*. In: The Shapley value. Essays in honor of Lloyd S. Shapley, pages 101–119. Cambridge University Press, 1988.

MATHEMATICS RESEARCH UNIT, FSTC, UNIVERSITY OF LUXEMBOURG, 6, RUE COUDENHOVE-KALERGI, L-1359 LUXEMBOURG, LUXEMBOURG
E-mail address: jean-luc.marichal[at]uni.lu

UNIVERSITY OF LIÈGE, DEPARTMENT OF MATHEMATICS, GRANDE TRAVERSE, 12 - B37, B-4000 LIÈGE, BELGIUM
E-mail address: p.mathonet[at]ulg.ac.be